Finite Plastic Deformation Due to Crystallographic Slip

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A general relationship between the amount of glide shear (due to slip) and the macroscopic shape change has been developed. Since the deformation can be large, finite strain analysis is employed. In this treatment, the shape change is expressed by a deformation gradient matrix, $F \equiv [\partial x_i/\partial X_j]$, *where* X *and* x *refer to the initial and final positions of a particle. This matrix is readily evaluated in terms of the amount of glide shear,* a, *and the direction cosines of the slip-plane normal and the slip direction for a single active slip system. In the case of deformation from slip on several systems, the product* F(a) *of the several deformation gradient matrices is first calculated. Then, by assuming that the final configuration is reached by a long series of small shears of magnitude* a, *occurring more or less alternately in the several slip systems, the final deformation gradient matrix can be obtained. Mathematically, the problem is reduced to finding the limit of* $F(a)^N$ *as* $N \rightarrow \infty$ *while* $a \rightarrow 0$ *in such a way that the product* $Na = \alpha$, *a finite constant designating the accumulated amount of shear. It turns out that this limit is simply* $e^{\alpha F_1}$, *where* F_1 *is the matrix whose elements are the coefficients of* a *in the* F(a) *matrix. Application is made of the present treatment to a fcc crystal of Permalloy compressed on the (110) plane and contrained to elongate in the [$\bar{1}$12] direction. In addition, it is shown that one may readily obtain from the general analysis the well-known formulas relating elongation, amount of glide shear, and amount of lattice rotation for crystals deforming by single and double slip under tension.*

IN problems of crystal plasticity, it is often necessary to relate the amount of glide in the operating slip systems to the macroscopic strain components. This relationship is relatively simple when only small strains are considered.[1-2] In this case the separate strain contributions from several slip systems are additive. Such a procedure is incorrect in the case of large plastic deformation. To the authors' knowledge, no general treatment of the latter problem has appeared in the literature. In specific instances, Mark, Polanyi, and Schmid[3] have derived the relationship between glide and axial elongation during tensile pulling of a single crystal when only one slip system is active. A similar relationship for duplex slip was worked out by v. Göler and Sachs.[4] Taylor and Elam[5] have also studied in detail problems of large plastic deformation. Due to uncertainties as to the slip systems at the time, however, they were mainly concerned with proving that slip occurs on {111}⟨110⟩ systems in fcc metals.

The present general treatment arose out of recent investigations of magnetic anisotropy in cold-worked Fe-Ni alloys[6] as well as some related problems in strength anisotropy of single crystals.[7] Detailed application of the general analysis will be provided in the case of a Permalloy single crystal compressed on the (110) plane and constrained to elongate only in the [$\bar{1}$12] direction. It will be seen that, as expected, the small strain approximation leads to significant errors after moderate straining. It will also be shown that the present general treatment yields the formulas of Mark *et al.* and of v. Göler and Sachs in those specific cases.

GENERAL CONSIDERATIONS[8]

Consider a homogeneous deformation in which a material point initially at (X_1, X_2, X_3) moves to (x_1, x_2, x_3), both positions being referred to the same set of Cartesian axes. Such a deformation can be

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specified completely by the deformation gradient matrix, F, of components $\partial x_i/\partial X_j$:

$$F \equiv \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \quad [1]$$

The displacement components are $u_i = x_i - X_i$, and hence the matrix [1] can be written in terms of the displacement derivatives by the substitutions

$$\frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \quad [2]$$

where δ_{ij} is the Kronecker delta.

In view of Eq. [2], the displacement derivatives and the small strain components

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \quad [3]$$

can readily be obtained from the matrix F.

Since the nine quantities $\partial x_i/\partial X_j$ specify the deformation completely, all quantities associated with the deformation can be derived from them. For example, the ratio of final volume to initial volume is the determinant of F:

$$\frac{V_{\text{final}}}{V_{\text{initial}}} = \det F \quad [4]$$

The ratio, $\lambda_{\mathbf{P}}$, of final length to initial length for any material line can be found from

$$\lambda_{\mathbf{P}}^2 = \sum_{i,j,k} \frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} P_j P_k \quad [5]$$

where $\mathbf{P}$ is the unit vector giving the initial direction of the line. The unit vector $\mathbf{p}$ giving the final direction of the line (after the deformation) has the components

$$p_i = \frac{1}{\lambda_{\mathbf{P}}} \sum_j \frac{\partial x_i}{\partial X_j} P_j, \qquad i = 1, 2, 3 \quad [6]$$

In general, the rotation of a material plane and the change in the perpendicular distance between parallel specimen faces is most conveniently expressed in terms of the quantities $\partial X_i/\partial x_j$, which may be obtained by inversion of the matrix F. The ratio $f_{\mathbf{Q}}$ of initial to final perpendicular distance between material planes of initial unit normal $\mathbf{Q}$ can be found from

$$f_{\mathbf{Q}}^2 = \sum_{i,j,k} \frac{\partial X_i}{\partial x_j}\frac{\partial X_k}{\partial x_j} Q_i Q_k \quad [7]$$

while the planes acquire the final unit normal $\mathbf{q}$ of components

$$q_j = \frac{1}{f_{\mathbf{Q}}} \sum_i Q_i \frac{\partial X_i}{\partial x_j} \quad [8]$$

If the initial and final normal can be identified as the same material line, then $f_{\mathbf{Q}} = 1/\lambda_{\mathbf{Q}}$.

An important advantage of specifying a deformation by its deformation gradient matrix is that the matrix for the resultant of two or more successive deformations is the product of the matrices for the individual deformations. For example, consider two successive deformations such that the first moves points initially at (X_1, X_2, X_3) to some intermediate configuration (y_1, y_2, y_3) specified by the deformation gradient matrix

$$F_A = \left[\frac{\partial y_i}{\partial X_j}\right] \quad [9]$$

while the second deformation, from the intermediate configuration (y_1, y_2, y_3) to the final configuration (x_1, x_2, x_3), is specified by

$$F_B = \left[\frac{\partial x_i}{\partial y_j}\right] \quad [10]$$

Since

$$\frac{\partial x_i}{\partial X_j} = \sum_{k=1}^{3} \frac{\partial y_k}{\partial X_j}\frac{\partial x_i}{\partial y_k}$$

the deformation gradient matrix for the resultant deformation from (X_1, X_2, X_3) to (x_1, x_2, x_3) is given by the matrix product $F_B F_A$, *i.e.*,

$$\left[\frac{\partial x_i}{\partial X_j}\right] = \left[\sum_{k=1}^{3} \frac{\partial x_i}{\partial y_k}\frac{\partial y_k}{\partial X_j}\right] = F_B F_A \quad [11]$$

Finally, although we have no occasion to use them in the present paper, it may be noted that the finite strain components in the material (or Lagrangian) description are the elements of $(1/2)(F^T F - I)$ while the corresponding quantities in the spatial (Eulerian) description are the elements of $(1/2)[I - (F^{-1})^T F^{-1}]$. Here I is the unit matrix and the superscript T denotes the transpose.

SINGLE SLIP

Consider a unit vector $\mathbf{m}$ (with components m_1, m_2, m_3) along the slip direction and a unit vector $\mathbf{n}$ (components n_1, n_2, n_3) normal to the slip plane. Then, with the origin considered a fixed point, we have

$$u_i = x_i - X_i = a(\mathbf{X}\cdot\mathbf{n})m_i \qquad i = 1, 2, 3 \quad [12]$$

where $\mathbf{X}$ is the position vector with components X_i, and a is the amount of simple shear resulting from slip. Upon noting that $\mathbf{X}\cdot\mathbf{n} = \sum_j X_j n_j$, we find, by differentiation of [12] in accordance with [2],

$$\frac{\partial x_i}{\partial X_j} = \delta_{ij} + a m_i n_j \quad [13]$$

or

$$F = \begin{bmatrix} 1 + am_1n_1 & am_1n_2 & am_1n_3 \\ am_2n_1 & 1 + am_2n_2 & am_2n_3 \\ am_3n_1 & am_3n_2 & 1 + am_3n_3 \end{bmatrix} \quad [14]$$

Eq. [14] can be written as

$$F = I + amn^T \quad [14a]$$

where m and n are the single-column matrices of direction cosines, and the superscript T denotes the transpose as before; *i.e.*, n^T is a single-*row* matrix. Since the slip direction $\mathbf{m}$ lies in the slip plane, *i.e.*, perpendicular to $\mathbf{n}$, we always have $\mathbf{m}\cdot\mathbf{n} = 0$. It follows that $\det F = 1$, *i.e.*, there is no volume change, and that F^{-1} for use in the general equations [7] and [8] is simply

$$F^{-1} = I - amn^T \quad [14b]$$

or